A smooth ride on a bumpy road

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1. Introduction

Washboarding is a pattern that occurs in unpaved roads as a result of vehicles driving above a critical speed. Such speed, dependent on the properties of the vehicles and the road surface, causes the formation of a series of bumps with short spacing. This phenomenon is not only aggravating for drivers but also poses as a hazard due to the fact that the adherence of the road is reduced. [3] A classic challenge that engineers face includes building an optimal vehicle that delivers a smooth car ride while maintaining cost efficiency. In this report, we develop two mathematical models that capture the dynamics of a vehicle driving on a washboard road and introduce filters to smooth or possibly eliminate the vehicle's vertical displacement due to the bumpy road. To gain insight on the behavior of the underlying system, we first consider a simplified model, the harmonic oscillator, the results of which we compare to a real world model. This real world model is an estimate of the underlying dynamics incorporated with a set of given data. The data obtained from both models shows that the smoothest ride is obtained by minimizing the displacement velocity of the vehicle.

Figure 1: Washboard pattern.

2. Derivation of the model

A number of assumptions are taken into consideration when deriving each model. We consider a one-wheeled car (i.e. a unicycle) in order to model the dynamics of driving over bumpy road followed by:

1. The suspension of a single wheel will be modeled as a damped harmonic oscillator.
2. The periodicity of the washboard road condition can be modeled as a sinusoidal forcing term. We consider this model with and without the presence of noise.
3. We assume the noise term is normally distributed with a known variance and zero mean.

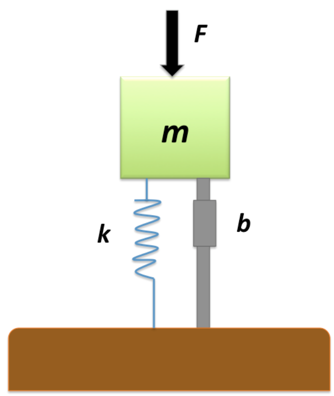

Figure 2

Under these assumptions, we appeal to Newton's Second Law of Motion: $\sum f = ma$, where $m > 0$ is the object mass and $a = y''(t)$ is the acceleration. For a damped harmonic oscillator this leads to the equation

$$my'' = -by' - \kappa y + f(t),$$

where $y(t) = y$ is the mass displacement, $f(t)$ is a forcing term due to the road, and $\kappa > 0$ and $b > 0$ are constants describing the stiffness of the spring and damping, respectively.

For simplicity, we assign $\beta = b/m \geq 0$, $\omega_0^2 = \kappa/m > 0$ and $f_m(t) = \frac{1}{m} f(t)$, leading to the following equation:

$$y'' + \beta y' + \omega_0^2 y = f_m(t).$$

Under the assumption of the washboard pattern road, we can replace $f_m(t)$ by $\alpha \sin(\omega_d t)$, obtaining:

$$y'' + \beta y' + \omega_0^2 y = \alpha \sin(\omega_d t), \tag{2.1}$$

where ω_0 and ω_d are the natural frequency and forcing frequency, respectively.

3. Mathematical Models

3.1 Model A: Harmonic Oscillator

As an accurate model of the dynamics we use a finite difference approximation to solve our system numerically. With a discrete uniform time mesh, t_n, we let $w^n = y(t_n)$, then using centered finite differences, $y''(t) \approx \frac{w^{n+1}-2w^n+w^{n-1}}{\Delta t^2}$, $y'(t) \approx \frac{w^{n+1}-w^{n-1}}{2\Delta t}$ then (2.1) becomes

$$\frac{w^{n+1} - 2w^n + w^{n-1}}{\Delta t^2} + \beta \left(\frac{w^{n+1} - w^{n-1}}{2\Delta t} \right) + \omega_0^2 w^n = f^n$$

$$\longrightarrow w^{n+1} = \left(\frac{2 - \omega_0^2 \Delta t^2}{1 + \frac{\beta \Delta t}{2}} \right) w^n + \left(\frac{-1 + \frac{\beta \Delta t}{2}}{1 + \frac{\beta \Delta t}{2}} \right) w^{n-1} + \left(\frac{\Delta t^2}{1 + \frac{\beta \Delta t}{2}} \right) f^n,$$

with $f^n = \alpha \sin(\omega_d t_n)$.

3.2 Model B: A Real World Model

It is rare to expect mathematical models to be perfect when they are compared to nature or complex engineered devices. In addition, it is rare to obtain measured data that is error free. For this reason, we take into consideration a modification of model A, which does not represent a damped harmonic oscillator. The dynamics for such a system is given by:

$$\begin{pmatrix} w^{n+1} \\ w^n \end{pmatrix} = \begin{pmatrix} (2 - \omega_0^2 \Delta t^2)\gamma & (1 - \frac{\beta \Delta t}{2})\gamma \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w^n \\ w^{n-1} \end{pmatrix} + \alpha \Delta t^2 \gamma \begin{pmatrix} \sin(\omega_d t_n) \\ 0 \end{pmatrix}$$

where $\gamma = (1 + \frac{\beta \Delta t}{2})^{-1}$.

For analysis purposes, it's sufficient to consider the convergence properties of the homogeneous case

$$w^{n+1} = (2 - \omega^2 \Delta t^2)\gamma w^n + \left(1 - \frac{\beta \Delta t}{2} \right) \gamma w^{n-1}, \tag{3.1}$$

where we make the ansatz of the form of solution to be $w^n = \lambda^n$.

Plugging this ansatz into Eq (3.1) gives

$$\begin{aligned} w^n &= a\lambda_1^n + b\lambda_2^n \\ &= a\left(\frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}\right)^n + b\left(\frac{c_1 - \sqrt{c_1^2 + 4c_2}}{2}\right)^n \end{aligned}$$

where $c_1 = (2 - \omega^2 \Delta t^2)\gamma$ and $c_2 = (1 - \beta\Delta t)\gamma$.

The following parameters and initial conditions are used to obtain numerical simulations throughout the report, unless otherwise specified: $\alpha = 0.06, \beta = 0.1, \omega = 0.7$. Using these values, we have the following solution to the homogeneous problem: $w_n = 0.0720[2.3552^n - (-0.4225)^n]$, which grows exponentially with respect to time and does not exhibit oscillatory behavior, as shown in Figure 3.

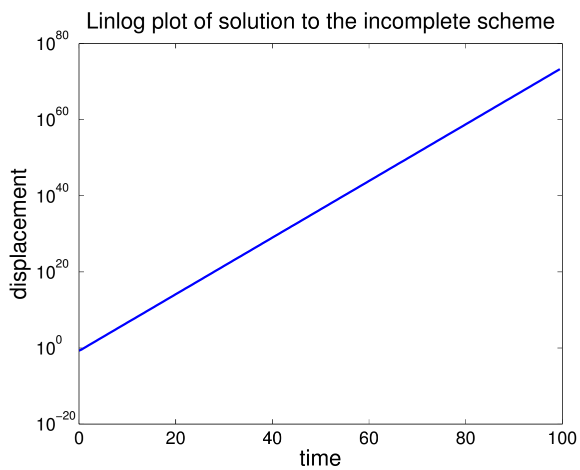

Figure 3: Displacement of Model A.

Note that this solution does not describe a simple harmonic oscillator, thus we should not expect it to behave as such.

4. Observation Data

To include the discomfort caused by the gravel road in addition to the washboard pattern, we assume that we have a sensor that gathers correct driving conditions at discrete uniform times. Additionally, we assume the observation data is subject to additive white noise with zero mean and known variance. The sampling data is chosen to be synchronous to the time step of the model. Our observation data at time t_m is Z_m, given below:

$$Z_m = Hy(t_m) + \eta_m$$

where H is some linear functional of the true conditions $y(t_m)$, with additive white noise η_m with known variance and zero mean. The observation data will provide useful feedback to obtain the smoothest ride possible.

4.1 Kalman Filtering

To enhance the smoothness of the ride, we introduce a technique known as the Kalman Filter which minimizes the variance between the measured data and the predicted data. We start by recalling Bayes theorem

$$\mathbb{P}(Y|Z) \propto \mathbb{P}(Z|Y)\mathbb{P}(Y)$$

which tells us our posterior distribution, $\mathbb{P}(Y|Z)$, is proportional to the product of the likelihood distribution and prior distribution, $\mathbb{P}(Z|Y)$, and $\mathbb{P}(Y)$.

By our assumption that the model and observation data includes white noise with known variance and zero mean, we can derive that the posteriori distribution is given by

$$\mathbb{P}(Y|Z) \propto \exp\left[-\sum \theta_n^\top Q^{-1}\theta_n - \sum \eta_m^\top R^{-1}\eta_m\right].$$

We wish to minimize the variance of the posteriori distribution.

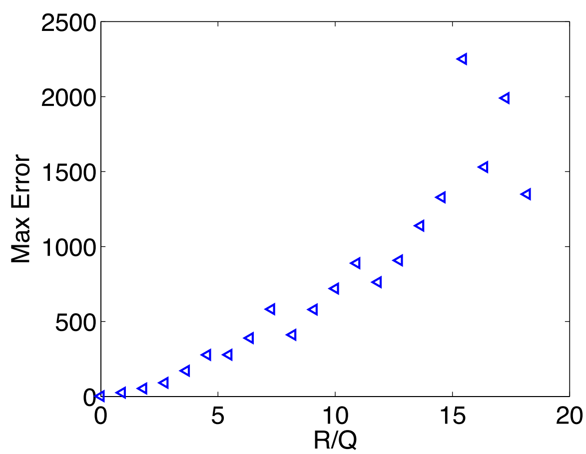

Figure 4: Kalman Filtering: Error between Kalman filter path and observation data. R: covariance of observation noise; Q: covariance of model noise.

Figure 4 shows the maximum difference between the model prediction and measured data as the ratio R/Q varies. One notices that when the ratio R/Q is small the Kalman filter will have higher confidence on the measured data. Conversely, as R/Q increases the estimated path will rely less on the measured data, and will adjust its behavior to be the best compromise between data and the model prediction.

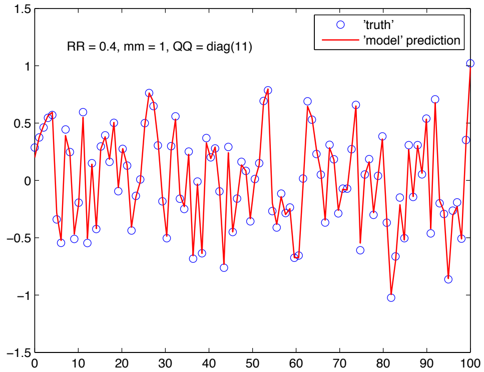

Figure 5: More confidence on the data.

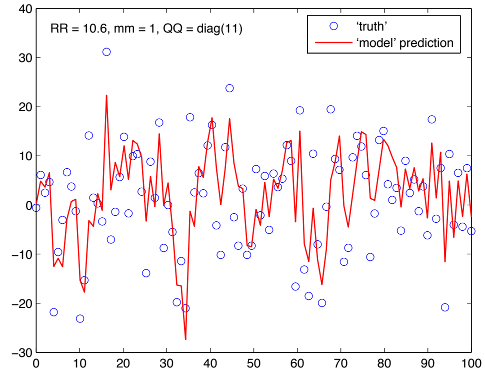

Figure 6: More confidence in the model.

5. Ride Comfort

To improve the quality of the ride for our passenger we introduce the concepts of passive and active filtering which serve to eliminate or reduce a combination of the displacement, velocity and the acceleration of the system. In passive filtering we implement this through physical modifications to the original parameters. In active filtering we provide a feedback loop using the Kalman filter.

5.1 The Passive Filter

In the preliminary attempt to smooth out the ride for our passenger we introduce a passive filter which acts in response to the washboard pattern of the road. Upon examination of the frequency spectrum of our displacement and velocity we introduce additional parameters g_a, g_v, g_d which manifest themselves as modifications to the mass, damping coefficient and spring constant of our system respectively. The inherent issue with this approach is that this method relies on physical modifications to the system which are dependent on the right hand side forcing function. Thus if the character of the road changes, the passive filtering needs to be manually adjusted which is inconvenient to do frequently.

Through numerical experiments, we observed that adjusting the spring constant by varying g_d leads to a shift of the natural frequency of the system which shifts the highest energy peak in the frequency spectrum (see fig. 7).

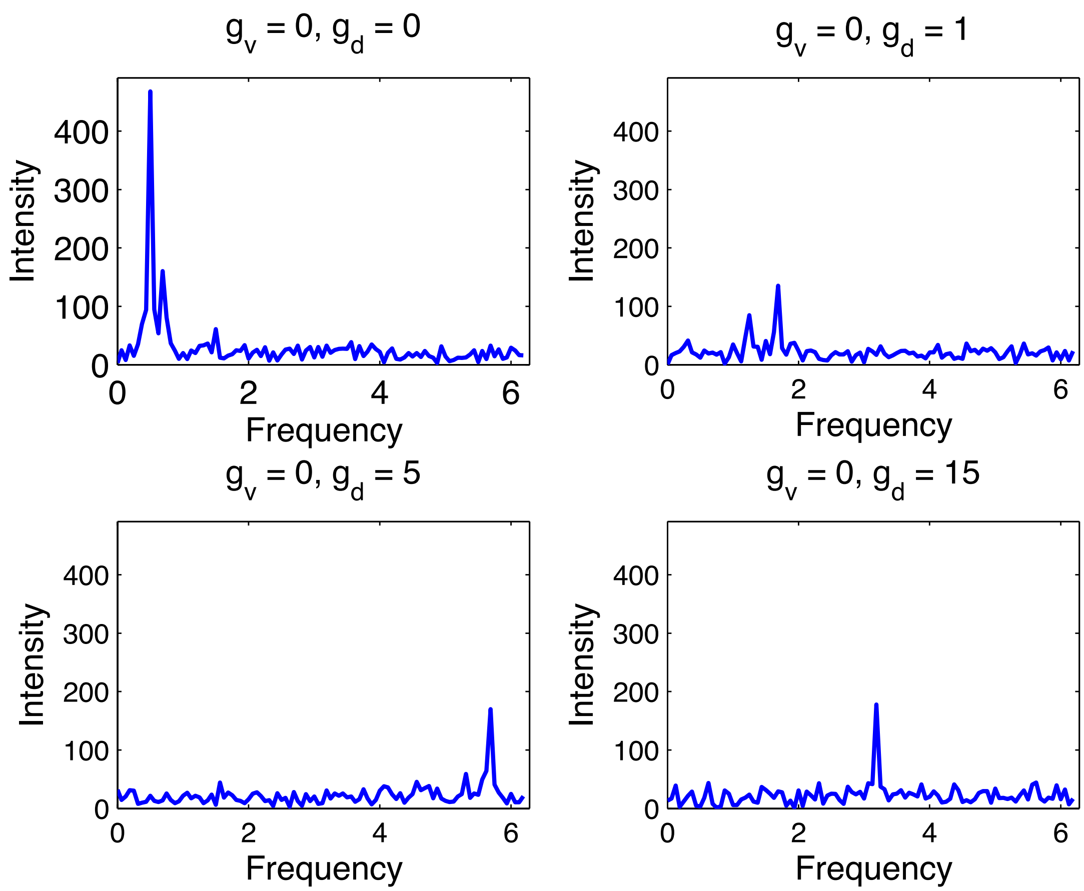

Figure 7: Passive Filtering: Varying g_d

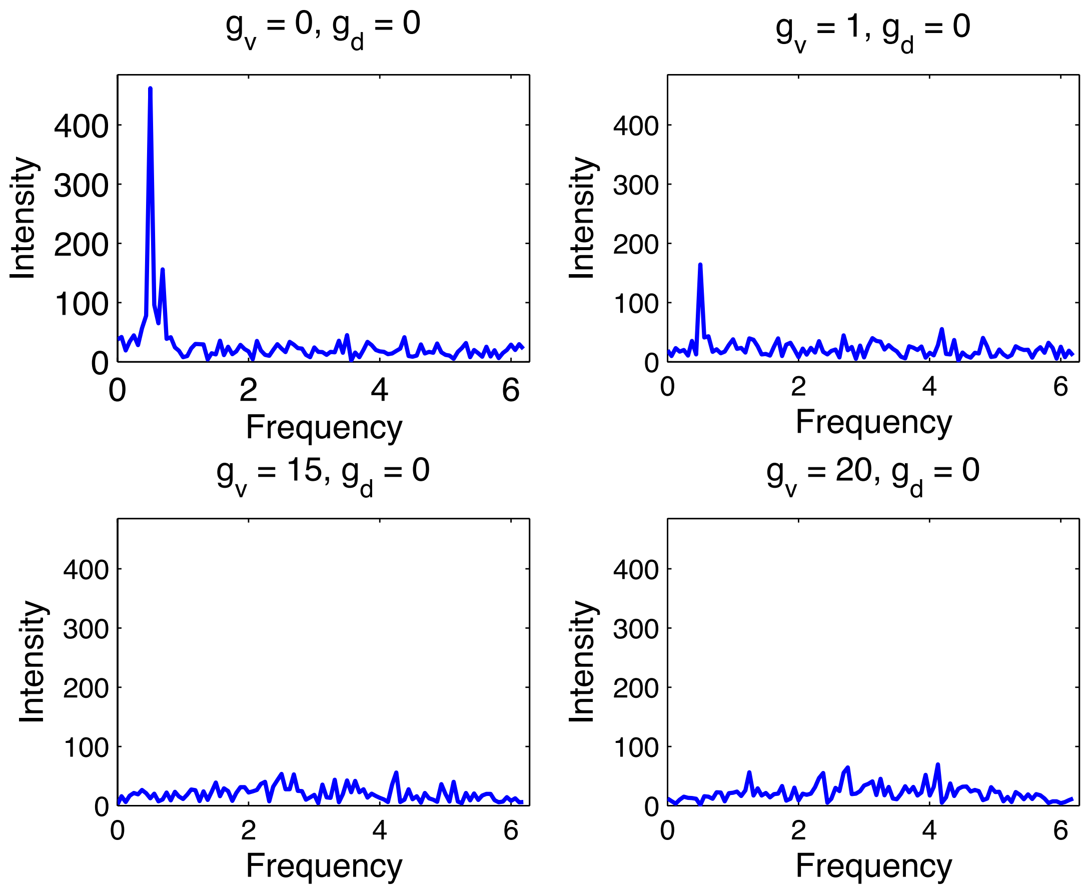

Figure 8: Passive Filtering: Varying g_v

On the other hand we found that adjusting the damping coefficient by varying g_v reduces the energy of the driving frequency. However as a consequence, this amplifies the effect of the white noise throughout the spectrum. This can be addressed for example through the use of a seat cushion.

5.2 The Active Filter

The best option for the active filtering is the use of the Kalman filter to reduce the bad effects of the bumpiness. The idea behind this methodology is to use the output from the Kalman filter combined with the passive filtering. Further details on Kalman filtering are in the Appendix.

6. Conclusions

We developed two mathematical models to describe the motion of a car on a bumpy road. Our first model is a benchmark to verify that the active and passive methods we used agree with expectations. The latter one represents a real world situation where one should not expect to know the actual dynamics of the system. Through the use of the Kalman filtering we corrected the inaccuracies of the model by introducing observation data. Using this new Kalman path we investigated ways to tune system parameters to provide a smooth ride on a bumpy road.

As future work one should include in this analysis a time delay which represents better an active control response system.

7. Appendix

Here we give a derivation of the Kalman filtering for discrete time and linear filtration problem. The derivation here follows lecture notes [2]. Derivation of Kalman filter for more complex settings can be found in [1].

Model and Data

We assume that the evolution of the system can be described by the following stochastic difference equation

$$x_{k+1} = M_k x_k + G_k u_k + \epsilon_k \qquad \text{(Model)} \tag{7.1}$$

where x_k is a vector which represents the state at time t_k, u_k is deterministic vector, ϵ_k is a white-noise (Markov-Gaussian) process with zero mean and covariance matrices

$$\mathbb{E}\{\epsilon_k \epsilon_l^T\} = Q_k \delta_{kl}$$

We also assume that we have data which is of the form

$$z_k = H_k x_k + \nu_k \qquad \text{(Data)} \tag{7.2}$$

where z_k is the data measured at time t_k and ν_k is a white-noise process with zero mean and covariance matrices

$$\mathbb{E}\{\nu_k \nu_l^T\} = R_k \delta_{kl}$$

We assume that the initial distribution of x_0, $\{\epsilon_k\}$ and $\{\nu_k\}$ are pairwise independent.

Prediction step

Let us consider the state vector x_{k+1} conditioned on the data (observation) $Z^k = \{z_1, \cdots, z_k\}$, we denote it by $x_{k+1|k} = x_{k+1}|Z^k$ and denote its minimum mean squared error estimator by $\hat{x}_{k+1|k} = \mathbb{E}\{x_{k+1}|Z^k\}$, we call it a prediction. By definition,

$$\begin{aligned} \hat{x}_{k+1|k} &= \mathbb{E}\{x_{k+1}|Z^k\} \\ &= \mathbb{E}\{M_k x_k + G_k u_k + \epsilon_k|Z^k\} \\ &= M_k x_{k|k} + G_k u_k \end{aligned} \tag{7.3}$$

Define $P_{k+1|k}$ the variance (or mean squared error) of $\hat{x}_{k+1|k}$,

$$P_{k+1|k} = \mathbb{E}\{(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T|Z^k\} \tag{7.4}$$

By using equations (7.1) and (7.3) and that fact that $\hat{x}_{k|k}$ and ϵ_k are independent, we have

$$\begin{aligned} P_{k+1|k} &= \mathbb{E}\{(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T|Z^k\} \\ &= M_k P_{k|k} M_k^T + Q_k \end{aligned} \tag{7.5}$$

Equation (7.3) and (7.5) are called the prediction step.

Analysis step

The idea of Kalman filtration is to use the data obtained from z_{k+1} to update the prediction $\hat{x}_{k+1|k}$ to get $\hat{x}_{k|k}$ such that $\hat{x}_{k|k}$ is the minimum mean square error estimator for $x_{k+1|k+1}$. We assume that $\hat{x}_{k+1|k+1}$ is linear combination of $\hat{x}_{k+1|k}$ and z_{k+1}, that is,

$$\hat{x}_{k+1|k+1} = K'_{k+1}\hat{x}_{k+1|k} + K_{k+1}z_{k+1}$$

for some matrices K'_k and K_k. For convenience, let us define the estimation error of $x_{k+1|k+1}$

$$\tilde{x}_{k+1|k+1} = \hat{x}_{k+1|k+1} - x_{k+1|k+1}$$

We assume that $\hat{x}_{k+1|k+1}$ is unbiased which requires

$$\mathbb{E}\{\hat{x}_{k+1|k+1}\} = K'_{k+1}\mathbb{E}\{\hat{x}_{k+1|k}\} + K_{k+1}H_{k+1}\mathbb{E}\{x_{k+1}\} \tag{7.6}$$

Since the prediction is unbiased

$$\mathbb{E}\{\hat{x}_{k+1|k}\} = \mathbb{E}\{x_{k+1}\}$$

we have

$$\mathbb{E}\{\hat{x}_{k+1|k+1}\} = (K'_{k+1} + K_{k+1}H_{k+1})\mathbb{E}\{x_{k+1}\}$$

We require that $K'_{k+1} + K_{k+1}H_{k+1} = I$ such that $\hat{x}_{k+1|k+1}$ is unbiased, hence

$$K'_{k+1} = I - K_{k+1}H_{k+1}$$

so now the estimate of x_{k+1} given Z^{k+1} is updated as

$$\begin{aligned} \hat{x}_{k+1|k+1} &= (I - K_{k+1}H_{k+1})\hat{x}_{k+1|k} + K_{k+1}z_{k+1} \\ &= \hat{x}_{k+1|k} + K_{k+1}(z_k - H_{k+1}\hat{x}_{k+1|k}) \end{aligned} \tag{7.7}$$

where we use the facts that v_{k+1} and $\tilde{x}_{k+1|k}$ are independent. we call the matrix K_k the Kalman gain.

Next we need to update the covariance matrix by using the data z_{k+1}. Recall the definition of $P_{k+1|k+1}$ and $P_{k+1|k}$, we have

$$\begin{aligned} P_{k+1|k+1} &= \mathbb{E}\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}|Z^{k+1}\} \\ &= (I - K_{k+1}H_{k+1})P_{k+1|k}(I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K^T_{k+1} \end{aligned} \tag{7.8}$$

Equations (7.7) and (7.8) are called the analysis step.

The last problem one needs to address is the choice of the Kalman gain. We want to choose a suitable Kalman gain K_{k+1} such that it minimizes mean square error, that is, we want to minimize

$$\begin{aligned} \mathbb{E}\{\|\tilde{x}_{k+1|k+1}\|^2|Z^{k+1}\} &= \mathbb{E}\{\tilde{x}^T_{k+1|k+1}\tilde{x}_{k+1|k+1}|Z^{k+1}\} \\ &= \text{trace}\mathbb{E}\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}|Z^{k+1}\} \\ &= \text{trace}(P_{k+1|k+1}) \end{aligned} \tag{7.9}$$

Taking the derivative of trace$(P_{k+1|k+1})$ with respect to $K_{k+1|k+1}$

$$\frac{\partial \text{trace}(P_{k+1|k+1})}{\partial K_{k+1}} = -2(I - K_{k+1}H_{k+1})P_{k+1|k}H^T_{k+1} + 2K_{k+1}R_{k+1} = 0 \tag{7.10}$$

Rearranging the terms of the above equation, we have

$$K_{k+1} = P_{k+1|k}H^T_{k+1}(H_{k+1}P_{k+1|k}H^T_{k+1} + R_{k+1})^{-1}.$$

We call this matrix Kalman gain and use it to update $\hat{x}_{k+1|k}$ to obtain $\hat{x}_{k+1|k+1}$.

References

[1] ANDREW H. JAZWINSKI, *Stochastic Processes and Filtering Theory*, Acaemic Press,

New York, 1970.

[2] http://www.robots.ox.ac.uk/ ian/Teaching/Estimation/LectureNotes2.pdf

[3] perso.ens-lyon.fr/nicolas.taberlet/washboard/